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CALCULATION OF THE ADJOINT MASSES FOR AN ANNULAR BLADE ASSEMBLY

L. A. Tkacheva

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It is necessary to know the adjoint-mass coefficients in order to solve various problems in turbine aeroelasticity such as the calculation of the natural frequencies and forms of blade vibrations. These coefficients are known only for the planar set of plates [1-3], so interest attaches to estimating the effects of the three-dimensional flow on their magnitudes.

Here we consider the adjoint masses for a three-dimensional ring set of thin blades performing small harmonic oscillations with a constant phase shift in an incompressible fluid.

We use a cylindrical coordinate system r, θ, z for the ring set of N blades vibrating in a liquid between two unbounded cylinders C_1 and C_2 with radii r_1 and r_2 (Fig. 1). We assume that: 1) The liquid is ideal and incompressible and is at rest at infinity, while the flow is potential; 2) the blades are infinitely thin and represent screw surfaces defined by the equations

$$z = h(\theta - 2\pi n/N),\ -\theta_0 < \theta - 2\pi n/N < \theta_0,\ r_1 < r < r_2,$$
$$n = 0,\ 1, \ldots, N-1,$$

where h the pitch of the screw surface and $2\theta_0$ the blade setting angle; 3) all the blades perform small oscillations with the same harmonic law but a certain phase shift $\mu = 2\pi n/N$, $n = 0, 1, \ldots, N-1$.

We transfer to dimensionless coordinates r', θ', z', referred to the characteristic length $L = r_2 - r_1$:

$$r' = r/L,\ z' = z/L,$$
$$\theta' = \theta,\ h' = h/L.$$

In what follows, the primes are omitted. By virtue of the third assumption, the vibration law can be put as

$$w^{(k)}(r, \theta, t) = Lf(r, \theta) \exp [i(k\mu + \omega t)],$$

where $w^{(k)}$ are the displacements of the points on blade k along the normal, ω is the circular frequency, and $f(r, \theta)$ is the dimensionless complex function that defines the form of the vibrations. We represent the velocity potential φ in the form

$$\varphi = iL^2\omega\Phi(r, \theta, z) \exp(i\omega t).$$

Here $\Phi(r, \theta, z)$ is a dimensionless complex function that satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{1}$$

and the following boundary conditions:

$$\lim_{|z|\to\infty} |\nabla\Phi| = 0; \tag{2}$$

$$\left.\frac{\partial\Phi}{\partial n}\right|_{S_k} = f(r, \theta) e^{ik\mu}, \quad k = 0, 1, \ldots, N-1; \tag{3}$$

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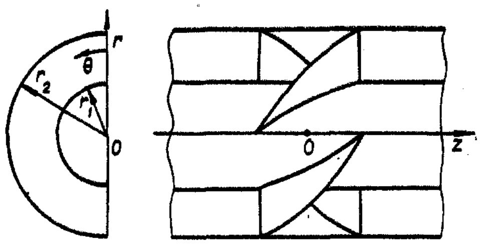

Fig. 1

$$\frac{\partial\Phi}{\partial n}\Big|_{C_j}=0, \quad j=1,2; \tag{4}$$

$$\Gamma=0, \tag{5}$$

where n is the normal to the corresponding surfaces, S_k is the surface of blade k, and Γ is the velocity circulation around a blade. By virtue of (3), the function Φ has the property of generalized periodicity:

$$\Phi(r,\ \theta+2\pi n/N,\ z)=e^{in\mu}\Phi(r,\ \theta,\ z),\ n=0,\ 1,\ \ldots,\ N-1. \tag{6}$$

The blades are represented as vortex surfaces of intensity γ, while the cylinders C_j are represented as simple-layer potentials with intensities $\varkappa_j$, j = 1, 2. The vector γ has components γ_r and γ_τ, where τ is the tangent to the screw line, and it is related to the velocity discontinued by

$$\boldsymbol{\gamma}=\mathbf{n}\times(\mathbf{v}^{+}-\mathbf{v}^{-}),$$

where the superscripts + and - denote the limiting values of the function on approach from above and below correspondingly. The components of γ satisfy the following [4]:

$$\partial(\gamma_r\sqrt{r^2+h^2})/\partial r+\partial\gamma_\tau/\partial\theta=0. \tag{7}$$

We assume that the radial component of the velocity does not have a discontinuity at the blade edges while the tangential component has an integrable singularity of order 1/2. Then the functions γ_r and γ_τ can be put as

$$\gamma_r(r,\ \theta)=\gamma_1(r,\ 0)/\sqrt{\theta_0^2-\theta^2},\ \gamma_\tau(r,\ \theta)=\gamma_2(r,\ \theta)\sqrt{\theta_0^2-\theta^2}. \tag{8}$$

From the conditions for low flow at the cylinders we have

$$\gamma_\tau(r_j,\ \theta)=0,\ j=1,2. \tag{9}$$

From the conditions for low flow at the blades we get the following system of three integral equations, of which the first is singular and the other two have weaker singularity:

$$i\omega Lf(r,\theta)=\sum_{n=0}^{N-1}\frac{e^{in\mu}}{4\pi\sqrt{r^2+h^2}}\iint_{S_0}\left\{\gamma_r\frac{\sqrt{\rho^2+h^2}\left[h^2(\psi-\theta)\cos\alpha_n+r^2\sin\alpha_n\right]}{\left[\rho^2-2\rho r\cos\alpha_n+r^2+h^2(\psi-\theta)^2\right]^{3/2}}-\right.$$
$$\left.-\gamma_\tau\frac{(\rho r+h^2\cos\alpha_n)(\rho-r\cos\alpha_n)-h^2r\sin^2\alpha_n+h^2\rho(\psi-\theta)\sin\alpha_n}{\left[\rho^2-2\rho r\cos\alpha_n+r^2+h^2(\psi-\theta)^2\right]^{3/2}}\right\}d\rho d\psi+$$
$$+\frac{1}{4\pi\sqrt{r^2+h^2}}\sum_{j=1}^{2}\iint_{C_j}\frac{\varkappa_j\left[r(\zeta-h\theta)-hr_j\sin(\psi-\theta)\right]r_jd\zeta d\psi}{\left[r_j^2-2r_jr\cos(\psi-\theta)+r^2+(\zeta-h\theta)^2\right]^{3/2}}; \tag{10}$$

$$\sum_{n=0}^{N-1}\frac{e^{in\mu}}{4\pi}\iint_{S_0}\frac{\gamma_r\sqrt{\rho^2+h^2}(h\psi-z)\sin\alpha_n+\gamma_\tau\rho\left[(h\psi-z)\cos\alpha_n-h\sin\alpha_n\right]}{\left[\rho^2-2\rho r_k\cos\alpha_n+r_k^2+(h\psi-z)^2\right]^{3/2}}d\rho d\psi+$$
$$+\frac{1}{4\pi}\sum_{j=1}^{2}\iint_{C_j}\frac{\varkappa_j\left[r_j\cos(\psi-\theta)-r_k\right]r_jd\psi d\zeta}{\left[r_j^2-2r_jr_k\cos(\psi-\theta)+r_k^2+(\zeta-z)^2\right]^{3/2}}+\frac{1}{2}(-1)\varkappa_k(\theta,z)=0,$$
$$k=1,\ 2. \tag{11}$$

Here $\alpha_n=\psi-\theta+2\pi n/N,\ n=0,\ 1,\ \ldots,\ N-1.$

We examine the asymptotes of the integrals in (11) for large z. We expand the integrand functions as series in negative powers of z and use the fact that for $\mu = 2m\pi/N$

$$\sum_{n=0}^{N-1} e^{in\mu} \cos^m \alpha_n = 0 \text{ and } \sum_{n=0}^{N-1} e^{in\mu} \sin \alpha_n \cos^{m-1} \alpha_n = 0,$$

which gives us that the first integral in (11) I_1 is of order $I_1 = O(z^{-\nu})$ for $|z| \to \infty$, where

$$\nu = \begin{cases} 2N, & \mu = 0, \\ \mu N/\pi, & 0 < \mu \leqslant \pi, \\ N(2 - \mu/\pi), & \pi < \mu < 2\pi. \end{cases} \tag{12}$$

Then one can see that $\varkappa_j$ has the same order at infinity:

$$\varkappa_j = O(z^{-\nu}) \text{ for } |z| \to \infty, \ j = 1, 2. \tag{13}$$

This means that the condition for the velocities to die away at infinity is met.

We put $f(r, \theta)$ in the form

$$f(r, \theta) = f_1(r, \theta) + f_2(r, \theta),$$

where the functions f_1 and f_2 satisfy the conditions

$$f_1(r, -\theta) = \overline{f_1(r, \theta)}, \ f_2(r, -\theta) = -\overline{f_2(r, \theta)}. \tag{14}$$

This representation enables one to solve the problems for f_1 and f_2 independently. Note that the function if_2 satisfies the first equation in (14). Therefore, it is sufficient to consider the problem for $f = f_1$. In that case, the solution satisfies the conditions

$$\gamma_r(r, -\theta) = -\overline{\gamma_r(r, \theta)}, \ \gamma_\tau(r, -\theta) = \overline{\gamma_\tau(r, \theta)}, \ \varkappa_j(-\theta, -z) = -\overline{\varkappa_j(\theta, z)}, \ j = 1,2. \tag{15}$$

Conditions (14) and (15) allow one to solve the problem only for $z > 0$, which considerably reduces the volume of computation.

Therefore, we have the system of equations (7), (10), and (11) with the conditions (5) and (13) to determine the unknown functions γ_r, γ_τ, $\varkappa_1$, and $\varkappa_2$. The pressure difference Δp across a blade can be represented in the linear approximation by means of a Cauchy-Lagrange integral in the form

$$\Delta p(r, \theta, t) = -i\rho\omega L e^{i\omega t} \int_{-\theta_0}^{\theta} \gamma_r \sqrt{r^2 + h^2}\, d\theta, \tag{16}$$

where ρ is the density of the liquid.

In accordance with (6)-(9) we seek an approximate solution in the form

$$\gamma_r \sqrt{r^2 + h^2} = i\omega L \sum_{m=1}^{N_1} g_m(r)\, m \cos m\sigma / \sin \sigma; \tag{17}$$

$$\gamma_\tau = i\omega L \theta_0 \sum_{m=1}^{N_1} g_m'(r) \sin m\sigma; \tag{18}$$

$$\varkappa_k = i\omega L \sum_{q=0}^{N_3} h_q^{(k)}(z)\, e^{i(qN+n)\theta}, \quad k = 1, 2, \tag{19}$$

where $g_m(r)$, $h_q^{(k)}(z)$ are dimensionless complex functions, $g'_m(r)$ is the derivative of g_m, $n = \mu N/2\pi$; N_1 and N_3 are integers, and σ is a new variable, which is related to θ by

$$\theta = \theta_0 \cos \sigma. \tag{20}$$

We approximate the functions $g_m(r)$ as follows. The interval $[r_1, r_2]$ is split up into N_2 equal parts, and in each part $[r_{k-1}, r_k]$ we represent the function $g_m(r)$ as a quadratic polynominal with complex coefficients:

$$g_m(r) = \frac{1}{2} a_{km} (r - r_{k-1})^2 + b_{km}(r - r_{k-1}) + c_{km}, \ r_{k-1} < r < r_k. \tag{21}$$

We specify smoothness in the functions $g_m(r)$, $g_m'(r)$ and obedience to (9), which allows us to eliminate $2N_1N_2$ coefficients a_{km}, b_{km}.

Then N_1N_2 coefficients remain unknown. The functions $h_q^{(k)}(z)$ are expanded in negative powers of z for $|z| > H$, where H is a sufficiently large number, or for $|z| < H$ they are represented as linear combinations of Chebyshev polynomials T_1:

$$h_q^{(k)}(z) = \begin{cases} z^{-\nu} \sum_{l=0}^{2} A_{ql}^{(k)} z^{-l}, & z > H, \\ \sum_{l=0}^{N_4} B_{ql}^{(k)} T_l(z/H), & |z| < H, \\ z^{-\nu} \sum_{l=0}^{2} C_{ql}^{(k)} z^{-l}, & z < -H, \end{cases}$$

where ν is defined by (12) and $A_{ql}^{(k)}, B_{ql}^{(k)}, C_{ql}^{(k)}$ are complex coefficients.

We specify smoothness in the functions $h_q^{(k)}(z)$ up to the second derivatives, and then the coefficients $A_{ql}^{(k)}$ and $C_{ql}^{(k)}$ can be expressed in terms of $B_{ql}^{(k)}$; as a result we get $2(N_3 + 1)(N_4 + 1)$ unknown complex coefficients. The solution of the form of (17)-(19) satisfies the conditions for the absence of circulation in the flow and for decay at infinity. We provide obedience to the no-flow conditions at the check points on the blades

$$r_k^* = (r_{k-1} + r_k)/2, \quad \sigma_j^* = (2j-1)\pi/2N_1,$$
$$k = 1, 2, \ldots, N_2, \; j = 1, 2, \ldots, [(N_1+1)/2]$$

and on the cylinders

$$\theta_l^* = \frac{\pi(2l+1)}{N(N_3+1)}, \quad z_q^* = H\cos(\pi q/N_4),$$

$$l = 0, 1, \ldots, N_3, \; q = 0, 1, \ldots, [N_4/2],$$

to get a system of linear algebraic equations for the unknown coefficients.

This algorithm was realized numerically with a BÉSM-6 computer. The main difficulty in calculating the matrix lies in calculating the double singular integrals. It can be shown that if the point (r^*, θ^*, z^*) does not lie on the boundary of a blade, these singular integrals as taken over the band $\Pi_\varepsilon = \{(r, \theta) : |\theta - \theta^*| < \varepsilon\}$ tend to zero for $\varepsilon \to 0$. Consequently, the double singular integral over a blade can be reduced to a repeated one. Here the integration with respect to ρ can be performed analytically by virtue of the representation of (21). As a result, the double singular integral in (10) reduces to the one-dimensional integral of the form

$$\int_0^\pi \left\{\frac{C_1(r^*)}{\psi - \theta} + C_2(r^*)\ln|\psi - \theta| + K(\psi - \theta)\right\}\cos m\sigma d\sigma,$$

where $C_1(r^*)$, $C_2(r^*)$ are the coefficients for the singularities and $K(\psi - \theta)$ is a regular function. The singular integrals in this expression are calculated analytically. As regards the regular integrals of the oscillating functions, these may be calculated by means of Filon's formulas [5].

From (16) and (17) we get for the pressure difference across the blade

$$\Delta p = \rho\omega^2 L^2 e^{i\omega t} \sum_{m=1}^{N_1} g_m(r)\sin m\sigma. \tag{22}$$

Therefore, the functions $g_m(r)$ define the load distribution.

Let the blade vibration law take the form

$$w(r, \theta, t) = L\sum_{k=1}^{N_0} q_k(t) f_k(r, \theta),$$

where $q_k(t)$ are dimensionless generalized coordinates and $f_k(r, \theta)$ are the forms of oscillation, while N_0 is the number of generalized coordinates.

The generalized hydrodynamic forces acting on a blade are calculated from

$$Q_k(t) = \iint_{S_0} \Delta p(r, \theta, t) f_k(r, \theta)\, ds. \tag{23}$$

TABLE 1

g_m \ r	1	1,25	1,5	1,75	2
$g_1(r)$	1,97	2,01	2,10	2,18	2,21
$g_2(r)$	0,13	0,13	0,14	0,14	0,14
$g_3(r)$	−0,04	−0,03	−0,001	0,04	0,07

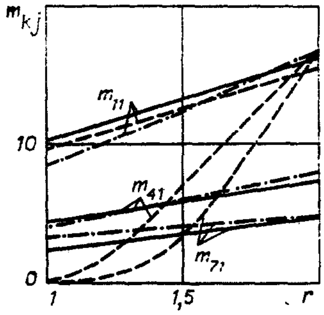

Fig. 2

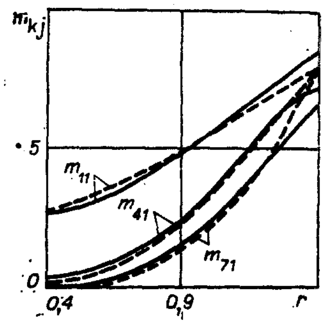

Fig. 3

We substitute (22) into (23) for the pressure difference to get

$$Q_k(t) = \rho L^4 \theta_0^2 \sum_{j=1}^{N_0} M_{kj} \ddot{q}_j(t),$$

where M_{kj} are the adjoint-mass coefficients, which in general are complex, but are real for $\mu = 0, \pi$.

We represent the adjoint masses in the form

$$M_{kj} = \int_{r_1}^{r_2} m_{kj}(r)\,dr.$$

Here $m_{kj}(r)$ are the lengthwise adjoint masses in the section r = const,

$$m_{kj}(r) = \sqrt{r^2 + h^2} \int_{-\theta_0}^{\theta_0} \Delta p(r, \theta, t) f_k(r, \theta)\,d\theta. \quad (24)$$

We examine the adjoint masses for the following forms of vibration:

$$f_k(r, \theta) = (r - r_1)^q \cos l\sigma, \; k = 1, 2, \ldots, 9, \; q, \; l = 0, \; 1, \; 2,$$

where $k = 3q + l + 1$, while θ and σ are related by (20). We substitute (22) for the pressure difference into (24) to get

$$m_{kj}(r) = \frac{\pi}{2} G_l^{(j)}(r) \sqrt{r^2 + h^2} (r - r_1)^q, \quad (25)$$

where $l = [k/3], \quad q = k - 1 - 3l,$

$$G_l^{(j)}(r) = \begin{cases} 2g_1^{(j)}(r), & l = 0, \\ g_2^{(j)}(r), & l = 1, \\ g_3^{(j)}(r) - g_1^{(j)}(r), & l = 2. \end{cases}$$

Here $g_l^{(j)}(r)$ are the pressure amplitude functions calculated for the vibration forms f_j.

As examples we considered two sets of blades with large and small elongations. The first set has the following parameters: $r_1 = 1, \; r_2 = 2, \; \theta_0 = 0{,}7. \; N = 6$, and in that case the elongation $\lambda = 0.39$, while the spacing and the angle of entry into the average section were correspondingly $\tau = 1.6, \; \beta = -56°$.

It was found that the functions g_m for the blades with small elongation were almost independent of radius, i.e., the pressure on the blade tends to equalize in scale. Table 1 gives the values of the $g_m(r)$ for the vibration form $f_4 = r - r_1$ and the phase shift $\mu = 2\pi/3$.

The adjoint masses can be calculated in that case by means of the theory of planar sets in the average section $r_0 = (r_1 + r_2)/2$ and for the vibrational law averaged with respect to radius:

$$f_0(\theta) = \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} f(r, \theta)\, dr,$$

and then they may be calculated in each section by means of (25). The same derivation was given for a straight set of plates in the case of small elongations and small Struhal numbers [6].

As an illustration, the solid lines in Fig. 2 give the values for the three adjoint-mass coefficients m_{11}, m_{41}, m_{71} as functions of radius, together with the corresponding values calculated in the hypothesis of cylindrical sections (broken line) and from the averaged vibration law (dot-dash line). The three curves agree well for m_{11} with the form of oscillation independent of the radius. In the cases of m_{41} and m_{71}, the forms of the vibration are linearly and quadratically dependent on the radius correspondingly. In that case, the cylindrical-section hypothesis gives a large deviation. A similar picture is observed for the upper adjoint-mass coefficients.

In the second case, the parameters were $r_1 = 0.4$, $r_2 = 1.4$, $\theta_0 = 0.74$, $h = 0.9$, $N = 30$ and $\lambda = 5.4$ with the corresponding values in the average section $\tau = 1$, $\beta = 45°$. In that case, the cylindrical-section hypothesis gives a better approximation to the results from the three-dimensional theory. Figure 3 shows the dependence of m_{11}, m_{41}, m_{71} on radius for $\mu = 2\pi/15$.

The convergence of the method was determined by numerical experiment by comparing the results with increasing values of H and $N_i (i = 1, \ldots, 4)$; it was found that for blades with small elongation, the results were almost independent of H. Even for $N_1 = N_2 = 4$, $N_3 = N_4 = 3$, any further change in N_i $(i = 1, \ldots, 4)$ altered the results only in the third figure.

For the blades with large elongation, the pressure varies substantially over them; therefore the convergence is somewhat worse. In that case, the number N_2 of divisions along the radius should be increased to provide sufficient accuracy. For example, for the case with $\lambda = 5$ and $\tau = 1$ with $N_1 = 4$, $N_2 = 6$, $N_3 = N_4 = 3$ any further increase in $N_i (i = 1, \ldots, 4)$ alters the results only in the third place.

In conclusion we note that the cylindrical-section hypothesis gives good results in calculating the adjoint masses for blades of large elongation, and it is also applicable to ones with small elongation if the form of the vibration is independent of the radius, and also that in the general case for blades with small elongation it is recommended to use the average-section method, which agrees well with the three-dimensional theory.

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